

Dirac equation in a de Sitter expansion for massive neutrinos from modern Kaluza-Klein theory.

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Abstract

Using the modern Kaluza-Klein theory of gravity (or the Induced Matter theory), we study the Dirac equation for massive neutrinos on a de Sitter background metric from a 5D Riemann-flat (and hence Ricci-flat) extended de Sitter metric, on which is defined the vacuum for test massless 1/2-spin neutral fields minimally coupled to gravity and free of any other interactions. We obtain that the effective 4D masses of the neutrinos can only take three possible values, which are related to the (static) foliation of the fifth and noncompact extra dimension.

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I. INTRODUCTION

In models based on supergravity [1], it has been pointed out the existence of some light particles whose interactions are suppressed at scales close to $M = M_P/(8\pi) \simeq 2.4 \times 10^{18}$ GeV. Such particles have nothing to do with the collider experiments, but may affect the standard scenario of big-bang cosmology [2–4]. The gravitino, which is the gauge field associated with local supersymmetry (SUSY), is one of the weakly interacting particles in supergravity models, and we expect the mass of the gravitino to be of the order of the typical SUSY-breaking scale. When the gravitino decays into a neutrino and a sneutrino, the emitted high energy neutrinos scatter off the background neutrinos and produce charged leptons (mainly electrons and positrons), which cause an electro-magnetic cascades and produce many soft photons. Hence, the propagation of neutrinos should be very important during inflation. Inflationary cosmology can be recovered from a 5D vacuum[5–7]. The inflationary theory is very consistent with current observations of the temperature anisotropy of the Cosmic Microwave Background (CMB)[8]. The most popular model of supercooled inflation is chaotic inflation[9]. In this model the expansion of the universe is driven by a single scalar field called inflaton. At some initial epoch, presumably the Planck scale, the scalar field is roughly homogeneous and dominates the energy density, which remains almost constant during all the inflationary epoch.

On the other hand inflation can be recovered from the Campbell-Magaard theorem[10–14], which serves as a ladder to go between manifolds whose dimensionality differs by one. This theorem, which is valid in any number of dimensions, implies that every solution of the 4D Einstein equations with arbitrary energy momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in vacuum. Because of this, the stress-energy may be a 4D manifestation of the embedding geometry. Physically, the background metric there employed describes a 5D extension of an usual de Sitter spacetime, which is the 4D spacetime that describes an inflationary expansion.

In this Letter we study the Dirac equation for 4D massive neutrinos in a de Sitter expansion using Modern Kaluza-Klein theory of gravity (or Induced Matter theory)[15, 16]. In this theory the 5D massless 1/2-spin test fields are considered free from interactions and minimally coupled to gravity on a 5D Ricci-flat metric in which we define the physical vacuum. Our approach is something different (but complementary) than the studied by Wesson

in [17], because we make a detailed study of the geometrical spinor properties of 5D vector fields that, once we make a static foliation on the fifth coordinate, can be considered as 4D massive neutrinos.

II. THE 5D CLIFFORD ALGEBRA AND SPINORS IN 5D

We consider the Ponce de León metric[18]

$$dS^2 = \left(\frac{\psi}{\psi_0} \right)^2 [dt^2 - e^{2t/\psi_0} dR^2] - d\psi^2. \quad (1)$$

The resulting 4D hypersurface after making $\psi = \psi_0$ describes a de Sitter spacetime. From the relativistic point of view an observer moving with the penta-velocity $U_\psi = 0$, will be moving on a spacetime that describes a de Sitter expansion which has a scalar curvature ${}^{(4)}R = 12/\psi_0^2 = 12H_0^2$, such that the Hubble parameter is defined by the foliation $H_0 = \psi_0^{-1}$. If we foliate $\psi = \psi_0$, we get the effective 4D metric

$$dS^2 \rightarrow ds^2 = dt^2 - e^{2H_0 t} d\vec{R}^2, \quad (2)$$

which describes a 3D spatially flat, isotropic and homogeneous de Sitter expanding universe with a constant Hubble parameter H_0 .

To define a 5D vacuum we shall consider a Lagrangian for a massless 5D spinor field minimally coupled to gravity (we shall consider $\hbar = c = 1$)

$$L = \frac{1}{2} [\bar{\Psi} \gamma^A (\nabla_A \Psi) - (\nabla_A \bar{\Psi}) \gamma^A \Psi] + \frac{R}{2\bar{K}}, \quad (3)$$

where $\bar{K} = 8\pi G$ and γ^A are the Dirac matrices which satisfy

$$\{\gamma^A, \gamma^B\} = 2g^{AB} \mathbb{I}, \quad (4)$$

such that the covariant derivative of the spinor Ψ on (1) is defined in the following form:

$$\nabla_A \Psi = (\partial_A + \Gamma_A) \Psi, \quad (5)$$

and the spin connection is given by

$$\Gamma_A = \frac{1}{8} [\gamma^b, \gamma^c] e_b^B \nabla_A [e_c B], \quad (6)$$

being $\nabla_A [e_c B] = \partial_A e_c B - \Gamma_{AB}^D e_c D$ the covariant derivative of the five-bein e_B^c (the symbol ∂_A denotes the partial derivative with respect to x^A and $\eta_{ab} = g_{AB} e_a^A e_b^B$ denotes the 5D

Minkowski spacetime in Cartesian coordinates), which relates the extended 5D de Sitter metric (1) with the 5D Minkowski spacetime written in Cartesian coordinates: $dS^2 = dt^2 - (dx^2 + dy^2 + dz^2) - d\psi^2$

$$e_B^c = \begin{pmatrix} \left(\frac{\psi}{\psi_0}\right) 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{\psi}{\psi_0}\right) e^{t/\psi_0} 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\psi}{\psi_0}\right) e^{t/\psi_0} 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{\psi}{\psi_0}\right) e^{t/\psi_0} 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

The Dirac matrices γ^a are represented in an Euclidean space instead of a Lorentzian space, and are described by the algebra[19, 20]: $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}, \quad (8)$$

such that $\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$, and the σ^i

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (9)$$

are the Pauli matrices.

A. Variables separation for the Dirac equation in 5D

Finally, using the fact that $\gamma^A = e_a^A \gamma^a$, we obtain the Dirac equation on the metric (1)

$$i \gamma^A \nabla_A \Psi = 0, \quad (10)$$

where we shall use (5) and (6) and Cartesian coordinates to describe the 3D Euclidean hypersurface. The Dirac equation (13) can be written as

$$i\gamma^a e_a^A \partial_A \Psi + \frac{i}{8} \gamma^a [\gamma^b, \gamma^c] e_a^A e_b^B g_{DB} (\partial_A e_c^D + \Gamma_{EA}^D e_c^E) \Psi = 0. \quad (11)$$

The relevant second kind Christoffel symbols are

$$\begin{aligned} \Gamma_{04}^0 &= \frac{1}{\psi}, \quad \Gamma_{11}^0 = \Gamma_{22}^0 = \Gamma_{33}^0 = \frac{e^{2\frac{t}{\psi_0}}}{\psi_0}, \quad \Gamma_{01}^1 = \Gamma_{02}^2 = \Gamma_{03}^3 = \frac{1}{\psi_0}, \\ \Gamma_{14}^1 &= \Gamma_{24}^2 = \Gamma_{34}^3 = \frac{1}{\psi}, \quad \Gamma_{00}^4 = \frac{\psi}{\psi_0^2}, \quad \Gamma_{11}^4 = \Gamma_{22}^4 = \Gamma_{33}^4 = -\frac{e^{2\frac{t}{\psi_0}}\psi}{\psi_0^2}. \end{aligned} \quad (12)$$

Hence, the Dirac equation on the 5D Riemann-flat metric (1) results to be

$$i \left\{ \gamma^0 \left[\left(\frac{\psi_0}{\psi} \right) \frac{\partial}{\partial t} + \frac{3}{2\psi} \right] + \left(\frac{\psi_0}{\psi} \right) e^{-\frac{t}{\psi_0}} [\vec{\gamma} \cdot \vec{\nabla}] + \gamma^4 \left[\frac{\partial}{\partial \psi} + \frac{2}{\psi} \right] \right\} \Psi = 0. \quad (13)$$

In order to make separation of variables we shall use the method introduced in [21]. After some algebraic manipulation, Eq. (13) can be rewritten as

$$(\hat{K}_{04} + \hat{K}_{123}) \Phi = 0, \quad (14)$$

where $\hat{K}_{04}\Phi = k\Phi = -\hat{K}_{123}\Phi$ and $\Phi = \gamma^0\gamma^4\Psi$ [By making $\det |k\Phi + \hat{K}_{123}\Phi| = 0$, we can evaluate the variable separation constant: $k = |\vec{k}|$]. The operators of separation \hat{K}_{04} and \hat{K}_{123} are given by

$$\hat{K}_{04} = \left[\gamma^0 e^{t/\psi_0} \left(\frac{\partial}{\partial t} + \frac{3}{2\psi_0} \right) + \frac{1}{\psi_0} \gamma^4 e^{t/\psi_0} \left(\psi \frac{\partial}{\partial \psi} + 2 \right) \right] \gamma^0 \gamma^4, \quad (15)$$

$$\hat{K}_{123} = \left[\gamma^1 \frac{\partial}{\partial x} + \gamma^2 \frac{\partial}{\partial y} + \gamma^3 \frac{\partial}{\partial z} \right] \gamma^0 \gamma^4, \quad (16)$$

where the condition $[\hat{K}_{04}, \hat{K}_{123}] = 0$ must be fulfilled. Since the metric (1) is 3D spatially isotropic, one obtains that the 3D spatial solutions can be expanded in term of harmonic functions, so that one can write

$$\Phi(t, x, y, z, \psi) \sim \Phi_0(t, \psi) e^{i\vec{k} \cdot \vec{x}}, \quad (17)$$

such that \vec{k} is the wavenumber of propagation on the 3D isotropic and homogeneous Euclidean space. Furthermore, one obtains

$$\hat{K}_{04}\Phi_0(t, \psi) = k \Phi_0(t, \psi). \quad (18)$$

After some algebra, we obtain that

$$(\hat{K}_0 + \hat{K}_4) \Upsilon(t, \psi) = 0, \quad (19)$$

such that

$$\hat{K}_4 \Upsilon = M \Upsilon = -\hat{K}_0 \Upsilon, \quad (20)$$

where we have used that $\Upsilon(t, \psi) = (\gamma^0)^{-1} \Phi_0(t, \psi)$, and

$$\hat{K}_0 = \left[\gamma^4 \left(\frac{\partial}{\partial t} + \frac{3}{2\psi_0} \right) - \mathbb{I} k e^{-t/\psi_0} \right] \gamma^0, \quad (21)$$

$$\hat{K}_4 = \left[\gamma^0 \frac{1}{\psi_0} \left(\psi \frac{\partial}{\partial \psi} + 2 \right) \right] \gamma^0, \quad (22)$$

that comply with $[\hat{K}_0, \hat{K}_4] = 0$. Using the first equation in (20), with the variables separation

$$\Upsilon(t, \psi) = \Upsilon_0(t) \Lambda(\psi), \quad (23)$$

we obtain the differential equation for $\Lambda(\psi)$

$$\frac{\psi}{\psi_0} \frac{\partial \Lambda}{\partial \psi} + \frac{2}{\psi_0} \Lambda = M \Lambda, \quad (24)$$

which is the same differential equation obtained in [22], but in a different framework. The solution for this equation is

$$\Lambda(\psi) = \Lambda_0 \left(\frac{\psi}{\psi_0} \right)^{M_0-2}, \quad (25)$$

where Λ_0 is a constant of integration and $M = M_0/\psi_0$ is a separation constant. For $M_0 < 2$ the function $\Lambda(\psi)$ tends to 0 for $\psi \rightarrow \pm\infty$, but is divergent for $\psi \rightarrow 0$. In order for the function $\Lambda(\psi)$ to be real, we must ask M_0 to take integer values and $M_0 \leq 2$: $M_0 = 2, 1, 0, -1, -2, -3, -4, \dots$. An interesting property is that for even $|M_0|$ values the function $\Lambda(\psi)$ is even but for odd $|M_0|$ values the function is also odd.

III. EFFECTIVE 4D DIRAC EQUATION FOR MASSIVE NEUTRINOS IN A DE SITTER SPACETIME

We assume a static foliation of the 5D spacetime on the 4D hypersurface Σ_0 , on which the 4D energy momentum tensor is described by a perfect fluid $\bar{T}_{\alpha\beta} = e_\alpha^A e_\beta^B T_{AB}|_{\psi_0} = (\rho + P) u_\alpha u_\beta - P h_{\alpha\beta}$, where $\rho(t, r, \psi_0)$ and $P(t, r, \psi_0)$ are the energy density and pressure of the

induced matter, respectively. The 4-velocities u_α are related to the 5-velocities U_A by $u_a = e_\alpha^A U_A$, and $h_{\alpha\beta} = e_\alpha^A e_\beta^B g_{AB}$ are the components of the metric tensor in (2). The Campbell-Magaard theorem, which is valid in any number of dimensions, implies that every solution of the 4D Einstein equations with arbitrary energy momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in vacuum. Because of this, the tensor $\bar{T}_{\mu\nu}$ is induced as a 4D manifestation of the embedding geometry.

If we take a constant foliation $\psi = \psi_0 \neq 0$ [to avoid a possible divergence of $\Lambda(\psi = \psi_0 = 1/H_0)$] on the metric (1) we obtain the metric(2), and the second equation in (20) takes the form

$$\left[\gamma^4 \gamma^0 \left(\frac{\partial}{\partial t} + \frac{3H_0}{2} \right) - \gamma^0 k e^{-H_0 t} \right] \Upsilon_0(t) = M \Upsilon_0(t), \quad (26)$$

where $M = M_0/\psi_0 = M_0 H_0$ is the induced mass of the neutrinos on the de Sitter spacetime (2). If we consider

$$\Upsilon_0(t) = \begin{pmatrix} \Upsilon_M^\uparrow(t) \\ \Upsilon_M^\downarrow(t) \end{pmatrix}, \quad (27)$$

such that $\Upsilon_M^\uparrow(t)$ and $\Upsilon_M^\downarrow(t)$ comply with the coupled differential equations

$$\frac{i}{c} \left(\frac{\partial}{\partial t} + \frac{3H_0}{2} \right) \Upsilon_M^\downarrow + (M - k e^{-H_0 t}) \Upsilon_M^\uparrow = 0, \quad (28)$$

$$-\frac{i}{c} \left(\frac{\partial}{\partial t} + \frac{3H_0}{2} \right) \Upsilon_M^\uparrow + (M + k e^{-H_0 t}) \Upsilon_M^\downarrow = 0. \quad (29)$$

One can work with both coupled Eqs. (28) and (29) in order to decouple $\Upsilon_M^\uparrow(t)$ and $\Upsilon_M^\downarrow(t)$, and obtain two decoupled second order differential equations

$$\begin{aligned} \frac{\partial^2 \Upsilon_M^\uparrow}{\partial t^2} &+ \frac{H_0 (3M + 4k e^{-H_0 t})}{(M + k e^{-H_0 t})} \frac{\partial \Upsilon_M^\uparrow}{\partial t} \\ &+ \left[\frac{3 H_0^2 k e^{-H_0 t}}{2(M + k e^{-H_0 t})} + \frac{9H_0^2}{4} - (M^2 - k^2 e^{-2H_0 t}) \right] \Upsilon_M^\uparrow = 0, \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial^2 \Upsilon_M^\downarrow}{\partial t^2} &+ \frac{H_0 (3M - 4k e^{-H_0 t})}{(M - k e^{-H_0 t})} \frac{\partial \Upsilon_M^\downarrow}{\partial t} \\ &+ \left[-\frac{3 H_0^2 k e^{-H_0 t}}{2(M - k e^{-H_0 t})} + \frac{9H_0^2}{4} - (M^2 - k^2 e^{-2H_0 t}) \right] \Upsilon_M^\downarrow = 0. \end{aligned} \quad (31)$$

The general solutions of Eqs. (30) and (31), are¹ ²

$$\begin{aligned}\Upsilon_M^{(\uparrow,\downarrow)}(t) &= e^{\mp\frac{i}{H_0}ke^{-H_0 t}} \left\{ C_1 e^{-\left[\frac{3}{2} + \frac{M}{H_0}\right]H_0 t} \mathcal{H}\text{eunC}[a i, a, -2, 0, 1, \mp x(t)] \right. \\ &\quad \left. + C_2 e^{-\left[\frac{3}{2} - \frac{M}{H_0}\right]H_0 t} \mathcal{H}\text{eunC}[a i, -a, -2, 0, 1, \mp x(t)] \right\},\end{aligned}\quad (32)$$

where $\mathcal{H}\text{eunC}[a i, a, -2, 0, 1, \mp x(t)]$ and $\mathcal{H}\text{eunC}[a i, -a, -2, 0, 1, \mp x(t)]$ are the Heun Confluent functions with arguments $\mp x(t) = \mp \frac{k}{M} e^{-H_0 t}$, and parameters $(a i, \mp a, -2, 0, 1)$. Here

$$a = 2 \frac{M}{H_0}.$$

Notice that for late times the Heun functions become

$$\mathcal{H}\text{eunC}[a i, a, -2, 0, 1, \mp x(t)]|_{H_0 t \gg 1} \rightarrow 1, \quad (33)$$

$$\mathcal{H}\text{eunC}[a i, -a, -2, 0, 1, \mp x(t)]|_{H_0 t \gg 1} \rightarrow \infty. \quad (34)$$

Furthermore, in this situation one obtains that

$$e^{\mp\frac{i}{H_0}k^{3/2}e^{-H_0 t}} \Big|_{H_0 t \gg 1} \rightarrow 1, \quad (35)$$

$$e^{-\left[9 + \sqrt{16\frac{M^2}{H_0^2} + 9}\right]\frac{H_0}{4}t} \Big|_{H_0 t \gg 1} \rightarrow 0. \quad (36)$$

In order for the spinors $\Upsilon_M^{(\uparrow,\downarrow)}$ to be well behaved for late times of inflation, the argument $\left[\frac{3}{2} + \frac{M}{H_0}\right]$ of the exponential $e^{-\left[\frac{3}{2} + \frac{M}{H_0}\right]H_0 t}$ must be positive or zero, and $C_2 = 0$. This requirement only fix a lower bound on M_0 , so that $-\frac{3}{2} \leq M_0 \leq 2$. There are three possible values of mass, which are

$$M_1 = 0, \quad M_2 = H_0, \quad M_3 = 2H_0. \quad (37)$$

This is a very strong result. Notice that if we make the extrapolation to present day values of the Hubble parameter, $\bar{H}_0 \simeq 10^{-33} H_0$, one would obtain that the present day masses of the neutrinos are

$$\bar{M}_1 = 0, \quad \bar{M}_2 = \bar{H}_0 \times 10^{-12} eV, \quad \bar{M}_3 = \bar{H}_0 \simeq 2 \times 10^{-12} eV, \quad (38)$$

¹ In the case of $M = 0$, both solutions $\Upsilon_{M=0}^{(\uparrow,\downarrow)}(t)$, are equal and the general solution takes the particular form

$$\Upsilon_{M=0}^{(\uparrow,\downarrow)}(t) = e^{-\frac{3}{2}H_0 t} \left\{ C_1 \sin \left[\frac{k}{H_0} e^{-H_0 t} \right] + C_2 \cos \left[\frac{k}{H_0} e^{-H_0 t} \right] \right\}.$$

² From the structure of the eq. (13) one can see that $\Upsilon_M^{(\uparrow,\downarrow)}$ are not Pauli spinors.

where we have taken $H_0 \simeq 10^{-9}G^{-1/2}$. These results are in agreement with evidence[23].

Finally, for very large times the solution (32) can be approximated to

$$\Upsilon_M^{(\uparrow,\downarrow)}(t) \Big|_{H_0 t \gg 1} \simeq C_1 e^{\mp \frac{i}{H_0} k e^{-H_0 t}} e^{-\left[\frac{3}{2} + \frac{M}{H_0}\right] H_0 t}. \quad (39)$$

where the spinors are normalized by $\langle \Psi | \Psi \rangle = 1$.

IV. FINAL REMARKS

We have studied the Dirac equation and solutions for effective 4D massive neutrinos on a de Sitter expansion, from a 5D Riemann-flat (and hence Ricci-flat), extended de Sitter metric. On this metric we have defined a 5D vacuum to test massless non-interacting fermion fields which are minimally coupled to gravity. After making a static foliation these fermions acquire an induced mass on the effective 4D curved de Sitter spacetime. However, the mass of the neutrinos can take only three possible values ($M_1 = 0$, $M_2 = H_0 = 1/\psi_0$ and $M_3 = 2/\psi_0$). If we consider present day values of the Hubble parameter the bigger mass \bar{M}_2 should be close to $\sim 2 \times 10^{-12}$ eV. This is a very strong result which assures that the effective 4D masses of the neutrinos are inversely proportional to the foliation, $\psi = \psi_0$, and shows how the mass of the neutrinos on a 4D de Sitter spacetime can be induced from a free massless 5D test spinors on a extended Riemann-flat (and hence Ricci-flat) metric which has non-zero connections Γ_{bc}^a . But more strong is the result that the masses of cosmological neutrinos should be dependent on the energy scale of the universe because they are dependent on the Hubble parameter. However, these results must be taken carefully because we have neglected any kind of interactions of the spinor fields with other fields. (These results are only valid for free 4D neutrinos that propagate freely in a 4D de Sitter background.) A more complete treatment will be studied in a future work.

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